20.1 Dynamic Strain From Waves

Elastic waves (P and S waves) at finite frequencies also contain information on the fault displacement and orientation as a function of time and distance. Consider a pattern of point sources that produce seismic waves and are recorded at a distance \( r \). How will they effect the displacement \( U \) at an observation point \((x_1, x_2, x_3)\)? We are seeking to solve the wave equation for the displacement \( \mathbf{u}(x, t) \) that satisfies the equation with a driving force related to the body forces on the plane, \( \mathbf{F} \), which is designated, \( x_0(t) \), and is applied in the \( x_1 \)-direction. The problem is known as Lamb's problem and is famous in the seismological literature as Lord Lamb first solved and published the solution in the late 1800's:

\[
\rho \ddot{\mathbf{u}} = \mathbf{F} + (\lambda + 2\mu) \nabla (\nabla \cdot \mathbf{u}) - \mu \nabla x (\nabla x \mathbf{u})
\]

Consider the geometry of a source, \( F \), that imparts a wave with a specific ray that emerges at the seismometer location to the right:

The point source, \( F \), is a vector with a unit force in the direction shown by the arrow.
Now defining direction cosines for the relationship between the direction of the force and the vector to the recording station as:

\[ \gamma_i = \frac{x_i}{r} = \frac{\partial r}{\partial x_i} \]

It turns out that we can also write the spatial derivatives of the distance as:

\[ \frac{\partial^2 r^{-1}}{\partial x_i \partial x_j} = (3\gamma_i \gamma_j - \delta_{ij}) r^{-3} \]

where \( x_0 \) is the time function from the source that starts at \( t = r/\alpha \); \( t' = (t - r / \alpha) \)

**Types of Sources:**

**A. Point sources:**

In an infinite, homogeneous media; operating in \( x_j \)-direction; the displacement in \( i^{th} \)-direction can be written as a combination the near-field and the far-field contributions:

\[
\vec{u}_i = \frac{1}{4\pi\rho} \left[ 3\gamma_i \gamma_j - \delta_{ij} \right] r^{-3} \int_{r(\rho)}^{r(\rho)} \frac{\partial t'}{t'} x_0(t - t') \, dt'
\]

\[ \text{Near-field contribution } \propto \frac{1}{r^3} \]

\( \gamma_i, \gamma_j \) - direction cosines between force applied and observation.

\[\vec{p} = \gamma_j \vec{A} + \gamma_j \vec{B} + \gamma_j \vec{C}\]
The far field contribution is given by:

\[
\gamma_i\gamma_j\left(\alpha^2 r\right)^{-1} x_o\left(\frac{t - r}{\alpha}\right) - \delta_{i\ j}\left(\beta^2 r\right)^{-1} x_o\left(\frac{t - r}{\beta}\right) \tag{20.1.2}
\]

where \( u_i^j \) is the wave field for the \( i^{\text{th}} \) direction along the \( j \)-axis.

**Far-field:** \( U^p \) (where \( r > \lambda \)). This term is of most interest in seismology because we are generally in the far-field with our seismographs and thus is considered for standard fault plane solutions.

**Displacements for Far-field P-waves:** For P waves the far-field displacement pattern, \( U^p \), is given by:

\[
u_i^p = \gamma_i\gamma_j\left(\alpha^2 r\right)^{-1} x_o\left(\frac{t - r}{\alpha}\right) / 4\pi\rho \tag{20.1.3}
\]

Conclusions:
1. The wave motion propagates with \( \alpha \).
2. The amplitude decays with \( 1/r \).
3. The waveform is proportional to the applied force, \( x_0 \).
4. The transverse component vanishes -- define vector \( \lambda'_i \) perpendicular to \( \gamma_i = \frac{x_i}{r_i} \)

so that \( \gamma_i\gamma_i = 0 \). Then \( \gamma'_i u_i^p \) is the transverse component of \( u^p \) in the \( \gamma' \) direction.

\[
\gamma'_i u_i^p = \left(\gamma'\gamma_i\right)\left(\gamma'_j\gamma_j\right)^{-1} x_o\left(\frac{t - r}{\alpha}\right) / 4\pi\rho = 0
\]

**Far-field S-wave, \( U^s \):**

\[
u_i^s = -\left(\gamma_i\gamma_j - \delta_{i\ j}\right)\left(\beta^2 r\right)^{-1} x_o\left(\frac{t - r}{\alpha}\right) / 4\pi\rho \tag{20.1.4}
\]

Conclusions:
1. The wave propagates with \( \beta \)
2. The amplitude decays with \( 1/r \)
3. The radial component (p-wave component) vanishes, where \( \gamma_iu_i^p \) is the radial component.

and

\[
\gamma_i u_i^p = - (\gamma_i \gamma_j - \gamma_j \delta_{ij}) (\beta^2 r)^{-1} \times o\left(t - \frac{r}{\beta}\right)/4\pi\rho = 0
\]

since \( \gamma_1 \gamma_i = 0 \); \( \gamma_1 \delta_{ij} = 0 \); resulting in \( \gamma_j - \gamma_j = 0 \)

Radiation Patterns of radial, \( u_p \); and transverse, \( u_s \):

\[
\begin{align*}
\gamma_j &= \text{cosine of angle between } x_j \text{ and force in } P \\
\gamma_j &= \text{cosine of angle between } x_j \text{ and force in } S
\end{align*}
\]

\[
\begin{align*}
\gamma_j u_j^p &= \gamma_j (\alpha^2 r)^{-1} x_o |t - x| / 4\pi\rho = \text{radiation pattern}
\end{align*}
\]

\[
\begin{align*}
\gamma_j u_j^s &= \gamma_j (\beta^2 r)^{-1} x_o |t - \beta| / 4\pi\rho
\end{align*}
\]

Where the relationship between the force vector F and ray is:

![Diagram](image-url)
Far field Radiation patterns: plots of energy or ground displacement versus azimuth, $\theta$.

**B. Dipolar Sources with Torque**

First, without torque

\[ \delta h, F_1, F_2 \]  
basic point sources - no torque  
$F_i$ - point forces

Dipole with torque

\[ \text{obs}(x_1, x_2, x_3) \]  
\[ \text{dipole with torque.} \]
Consider the P wave for a single point source in the $x_1$-direction, first:

$$u_1^p = \gamma_1 \gamma_1 \left( \frac{\alpha^2 r}{4\pi \rho} \right) x_0 \left( t - \sqrt{x_1^2 + (x_2 - \delta h_2)^2 + x_3^2} \right)$$

The displacement is for a total of two forces (includes $\delta h$, effect of the separation of the forces)

$$u_1^p = \frac{\partial u_1^p}{\partial h_2} = -\gamma_2 \gamma_1 \frac{\partial h_2}{4\pi \alpha^3 r} x_0 \left( t - \frac{r}{\alpha} \right)$$

Defining moment rate, $\dot{m}$

$$\lim_{\delta h_2 \to 0} \frac{\partial x_o}{\partial t} = \frac{\partial h_2}{\partial t} \quad x' = \dot{m} \quad \text{moment rate}$$

$$u_1^p = -\gamma_2 \gamma_1 \frac{\dot{m}}{4\pi \alpha^3} \frac{t - r/\alpha}{r} \quad \text{from } \gamma_1 \gamma_2, \quad u_1^p \text{ has a maxima at } \pm 45^\circ$$

Which has a radiation pattern:

![Radiation Pattern](image)
For S-waves:

But this pattern is not observed in actual earthquakes and a single couple with torque is unstable! i.e. it does not obey the requirement for stress equilibrium. So what have we left out?

C. **Double-Couple Without Moment**

Now consider the superposition of two orthogonal couples that gives zero net torque

P-waves

\[
\begin{align*}
\text{P-waves} & \\
\begin{array}{c}
\text{1 couple} \\
\text{1 couple} \\
\text{double couple} \\
\text{force equivalent}
\end{array}
\end{align*}
\]
Also define the **seismic moment**, \( M_o \), as the product of the fault area, \( A \), the displacement, \( \ddot{u} \), and the shear modulus, \( \mu \).

\[
M_o = \mu \ddot{u} A
\]

Now for a net effect of a force: \( x_o \) we see that:

\[
x_o \partial h_2 = \mu \ddot{u} A = M_o \text{ (seismic moment)}
\]

where \( x_o \partial h_2 \) is the earthquake moment, and \( \ddot{u} A \) is the force dislocation equivalent.

Physical interpretation (physics) of the faulting process:
Stress is related to strain by: \[ \frac{F}{A} = \mu \frac{\dd u}{\dd y} \approx \sigma_{12} = \mu \tau_{12} \] and \( dM_o = F \delta y = \mu \dd u A \).

Thus to determine the total moment we integrate over the entire fault surface of area \( A \):

\[
\int dM_o = \mu \int_A \dd u A \text{ resulting in } M_o = \mu \int \dd u A \quad \dd u - \text{average slip}
\]

or

\[ M_o = \mu \dd u A \quad \text{as before.} \]