17.2 Least-Squares Determination of Hypocenters

This is a mathematical formulation of a linear regression on four unknown variables of the earthquake epicenter \((x_0, y_0, z_0, t_0)\). The method is applicable especially when there are more observations of arrival times, \(m\), such as P or S-phases than epicenter parameters, \(n\), giving an overdetermined system of equations relating the travel times to the epicenter. This means that we have more stations and/or more phases for the event than the four unknown epicenter parameters.

Defining:

- \(x_0, y_0, z_0\) = the hypocenter coordinates
- \(t_0\) = origin time (GMT)
- \(x_i, y_i, z_i\) = coordinates of the \(i^{th}\) seismograph station
- \(P_i\) = the observed first P-wave arrival time at the \(i^{th}\)-station
- \(T_i\) = computed or theoretical travel time of the P-wave (or S-wave) based either upon an observed travel-time curve or computed by ray tracing.
- \(t_i\) = computed P- or S-wave arrival time at the \(i^{th}\) station, generally determined by a forward ray tracing algorithm.

Then the theoretical arrival time, \(t_i\), is the sum of the origin time, \(t_0\), and the travel time, \(T_i\):

\[
t_i = t_0 + T_i
\]  
17.2.1

Now we define the difference between the observed arrival time and the theoretical arrival time (for an initial or provisional hypocenter) as the residual difference, \(F_i\).

\[
F_i = P_i - t_i \text{ which is the arrival time anomaly or residual at the } i^{th} \text{ station.} 
\]  
17.2.2

Since \(t_i\) is a function of the three hypocenter coordinates \((x_0, y_0, z_0)\) and the origin time, \(t_0\), we can express the small change in computed arrival time, \(t_i\), due to small changes in these parameters by taking the total differential of \(dt_i\). Equivalently we can expand the differential, \(dt_i\), in a Taylor's series about the starting parameters \((x_0, y_0, z_0\) and \(t_0)\). Thus \(dt_i\) is defined as:

\[
dt_i = \frac{\partial t_i}{\partial x_0} dx_0 + \frac{\partial t_i}{\partial y_0} dy_0 + \frac{\partial t_i}{\partial z_0} dz_0 + \frac{\partial t_i}{\partial t_0} dt_0
\]  
17.2.3

which are the first-order terms of the Taylor's expansion.

Now define a new variable representing the error of the estimate as \(E_i\):

\[
E_i = F_i + dF_i
\]  
17.2.4

and differentiating 17.2.2, gives:

\[
dF_i = -dt_i
\]  
17.2.5
and we can now write:

\[ F_i - dt_i = E_i \]  \hspace{1cm} 17.2.6

where \( E_i \) is the arrival time anomaly after adjustment of \( dt_i \);

Hence, we can rewrite the arrival time error as:

\[ -E_i = \frac{\partial t_i}{\partial x_o} d x_o + \frac{\partial t_i}{\partial y_o} d y_o + \frac{\partial t_i}{\partial z_o} d z_o + \frac{\partial t_i}{\partial t_o} d t_o - F_i \]  \hspace{1cm} 17.2.7

But since the travel time is a function of the hypocenter coordinates and the origin time,

\[ t_i = T_i (x_o, y_o, z_o) + t_o \]  \hspace{1cm} 17.2.8

we can also write the following as the travel-time differentials that are linearly related to the origin time for each partial derivative:

\[ \frac{\partial t_i}{\partial x_o} = \frac{\partial T_i}{\partial x_o}, \quad \frac{\partial t_i}{\partial y_o} = \frac{\partial T_i}{\partial y_o}, \quad \frac{\partial t_i}{\partial z_o} = \frac{\partial T_i}{\partial z_o}. \]  \hspace{1cm} 17.2.9

Note that the derivative of the travel time with respect to the origin time is just unity, i.e. a change in one second of origin time corresponds exactly to a change of one second of travel time, therefore: \( \frac{\partial t_i}{\partial t_o} = 1 \)

Hence we can rewrite 17.2.7 as:

\[ \frac{\partial T_i}{\partial x_o} d x_o + \frac{\partial T_i}{\partial y_o} d y_o + \frac{\partial T_i}{\partial z_o} d z_o + \frac{\partial t_i}{\partial t_o} d t_o - F_i = E_i \]  \hspace{1cm} 17.2.10

where \( d x_o, d y_o, d z_o, d t_o \) are now the corrections to be made to the provisional or initial hypocenter to bring it closer to the true hypocenter.

How do these partial derivatives relate to the observed travel-time parameters? Using the chain rule, we define Geiger's coefficients as:

\[ \frac{\partial T_i}{\partial x_o} \frac{\partial \Delta_i}{\partial x_o \Delta = \Delta_i} = \alpha_i \]  \hspace{1cm} 17.2.11

\[ \frac{\partial T_i}{\partial y_o} \frac{\partial \Delta_i}{\partial y_o \Delta = \Delta_i} = \beta_i \]  \hspace{1cm} 17.2.12

\[ \frac{\partial T_i}{\partial z_o} \frac{\partial \Delta_i}{\partial z_o \Delta = \Delta_i} = \gamma_i \]  \hspace{1cm} 17.2.13
Note that the first part of the partial derivatives is just the inverse of the slope of the travel-time curve at a particular $\Delta_i$, i.e. the inverse of velocity or the slowness. The second term is a geometric relationship between $\Delta$ and the $x$, $y$ and $z$ coordinates. The Geiger coefficients are calculated for a given velocity model numerically from these expressions.

Defining for brevity; $dx_0 = y_1$, $dy_0 = y_2$, $dz_0 = y_3$, $dt_0 = y_4$

Equation 17.2.17 in this shorthand form can be written:

$$\alpha_i y_1 + \beta_i y_2 + \gamma_i y_3 + y_4 - F_i = -E_i.$$  

We now wish to calculate the adjustments, $Y_i$, in this overdetermined system of equations so that the sum of the residuals squared, $\Sigma E_i^2$, is a minimum, i.e. we will employ the least-squares criteria where the normal equations for a least squares minimum are given by differenting $E_i$ w.r.t. $y_j$ and setting it equal to zero:

$$\frac{\partial}{\partial y_j} \sum_{i=1}^{n} E_i^2 = 0,$$  

or

$$\sum_{i=1}^{n} E_i \frac{\partial E_i}{\partial y_j} = 0,$$  

for the $j = 1, 2, 3, 4$ unknown parameters

Now from 17.2.17, we can define the Geiger coefficients as:

$$\frac{\partial E_i}{\partial y_1} = -\alpha_i, \quad \frac{\partial E_i}{\partial y_2} = -\beta_i, \quad \frac{\partial E_i}{\partial y_3} = -\gamma_i, \quad \frac{\partial E_i}{\partial y_4} = -1$$

and expanding 17.2.17 using 17.2.18. As an example for the $j = 1$ equation this gives;

$$\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 + \ldots + \alpha_n y_n = 0.$$  

Then expressing $E_i$'s in terms of $y_i$'s by equation 17.2.17 for $j = 1$ we obtain a set of simultaneous equations:

$$\alpha_1 \alpha_1 y_1 + \alpha_1 \beta_1 y_2 + \alpha_1 \gamma_1 y_3 + \alpha_1 y_4 - \alpha_1 F_1 = 0$$

$$+ \alpha_2 \alpha_2 y_1 + \alpha_2 \beta_2 y_2 + \alpha_2 \gamma_2 y_3 + \alpha_2 y_4 - \alpha_2 F_2 = 0$$

$$\vdots$$

$$+ \alpha_n \alpha_n y_1 + \alpha_n \beta_n y_2 + \alpha_n \gamma_n y_3 + \alpha_n y_4 - \alpha_n F_n = 0.$$
Now summing columns and employing our summation convention for repeated indices and expanding 17.2.2 for \( j = 2, 3, \) and \( 4, \) we obtain:

\[
\begin{align*}
\text{\( j = 1 \)} & \quad \alpha_i \alpha_i y_1 + \alpha_i \beta_i y_2 + \alpha_i \gamma_i y_3 + \alpha_i \gamma_i y_4 - \alpha_i F_i = 0 \\
\text{\( j = 2 \)} & \quad \beta_i \alpha_i y_1 + \beta_i \beta_i y_2 + \beta_i \gamma_i y_3 + \beta_i \gamma_i y_4 - \beta_i F_i = 0 \\
\text{\( j = 3 \)} & \quad \gamma_i \alpha_i y_1 + \gamma_i \beta_i y_2 + \gamma_i \gamma_i y_3 + \gamma_i \gamma_i y_4 - \gamma_i F_i = 0 \\
\text{\( j = 4 \)} & \quad \alpha_i y_1 + \beta_i y_2 + \gamma_i y_3 + \gamma_i y_4 - F_i = 0
\end{align*}
\]

for \( i = 1, \ldots, n \) \hspace{1cm} 17.2.21

A further abbreviation can be made by defining:

\[
\begin{align*}
A_{11} &= \alpha_i \alpha_i & A_{12} &= \alpha_i \beta_i & A_{13} &= \alpha_i \gamma_i & A_{14} &= \alpha_i \\
A_{21} &= \beta_i \alpha_i & A_{22} &= \beta_i \beta_i & A_{23} &= \beta_i \gamma_i & A_{24} &= \beta_i \\
A_{31} &= \gamma_i \alpha_i & A_{32} &= \gamma_i \beta_i & A_{33} &= \gamma_i \gamma_i & A_{34} &= \gamma_i \\
A_{41} &= \alpha_i & A_{42} &= \beta_i & A_{43} &= \gamma_i & A_{44} &= 1
\end{align*}
\]

and

\[
B_1 = \alpha_i F_i \quad B_2 = \beta_i F_i \quad B_3 = \gamma_i F_i \quad B_4 = F_i \quad \text{all for } i = 1, \ldots, n. \hspace{1cm} 17.2.22
\]

Then equations 17.2.21 are then written in matrix form.

\[
\begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
A_{21} & A_{22} & A_{23} & A_{24} \\
A_{31} & A_{32} & A_{33} & A_{34} \\
A_{41} & A_{42} & A_{43} & A_{44}
\end{bmatrix}
\begin{bmatrix}
Y_1 \\
Y_2 \\
Y_3 \\
Y_4
\end{bmatrix}
= \begin{bmatrix}
B_1 \\
B_2 \\
B_3 \\
B_4
\end{bmatrix}
\hspace{1cm} 17.2.23
\]

Or

\[
\hat{\mathbf{X}} \hat{\mathbf{Y}} = \hat{\mathbf{B}} \hspace{1cm} 17.2.24
\]

We can now determine the solution vector, \( \hat{\mathbf{Y}}, \) through right multiplication by: \( \hat{\mathbf{X}}^{-1}. \)

\[
\hat{\mathbf{Y}} = \hat{\mathbf{X}}^{-1} \hat{\mathbf{B}} \hspace{1cm} 17.2.25
\]

or
The adjustments $\vec{Y}$ are then applied to the initial (or last calculated) hypocenter to get a new or corrected hypocenter which is the equivalent to the adjusted hypocenter made by applying the corrections:

$$\vec{Y} = (dx_o, dy_o, dz_o, dt_o)$$

$\vec{Y}$

17.2.26

Where the superscript 1 now refers to the previously estimated hypocenter. The procedure is then repeated using the last calculated hypocentral parameters, $x_0$, $y_0$, $z_0$, $t_0$, as the starting parameters for the next iteration. The iterations are terminated after the corrections $\vec{Y}$ become negligible (within tolerances set by the user) or if the equation 17.2.24 is not solvable.

We set the iteration variables on such parameters as RMS errors on each variable, azimuthal gap, distance from epicenter, etc.. For local earthquake arrays an RMS error for the time residuals should not exceed the standard deviation of the picking accuracy taking into consideration the error introduced in velocity structure which is necessary to determine the forward travel-time solutions.

For our use in local earthquake surveys, we usually chose the RMS error threshold as less than 0.4 sec. For regional earthquakes recorded across our regional network we chose an RMS error threshold at 0.8 sec.

There are various reasons for errors in the solutions, such as the assumption of one-dimensional velocity models when the earth is three-dimensionally lateral variable, because of local site effects where there are large lateral heterogeneities beneath the seismometers (this usually occurs for the upper 2 km), because of picking errors, or mis-identification of phases.