16.1 **Mathematical Definition of Dispersion:**

Consider two waves of slightly different phase velocities, $c_1$ and $c_2$, and wave numbers, $k_1$ and $k_2$, propagating in an elastic material. Figure 16.1 shows the conditions under which wave shapes can remain constant during propagation. Consider the amplitudes of each wave as sinusoids:

$$ y_1 = A \sin (k_1 x - \omega_1 t) \quad \text{where: } k_1 = \frac{\omega_1}{c_1} $$ \hspace{1cm} 16.1.1

and

$$ y_2 = A \sin (k_2 x - \omega_2 t) \quad k_2 = \frac{\omega_2}{c_2} $$ \hspace{1cm} 16.1.2

The amplitude, $Y$, of the wave is the sum of the two individual waves:

$$ Y = y_1 + y_2 = A \sin (k_1 x - \omega_1 t) + A \sin (k_2 x - \omega_2 t) $$

Because any analytic function can be written as the sum of its odd and even parts, the wave numbers and frequencies can be re-written as:

$$ k_1 = \frac{1}{2} [k_1 + k_2] + \frac{1}{2} [k_1 - k_2], $$

and

$$ k_2 = \frac{1}{2} [k_1 + k_2] - \frac{1}{2} [k_1 - k_2], $$ \hspace{1cm} 16.1.3

and

$$ \omega_1 = \frac{1}{2} [\omega_1 + \omega_2] + \frac{1}{2} [\omega_1 - \omega_2] $$

and

$$ \omega_2 = \frac{1}{2} [\omega_1 + \omega_2] - \frac{1}{2} [\omega_1 - \omega_2], $$ \hspace{1cm} 16.1.4

Defining the average of the wavenumbers, $k_0$, and frequencies, $\omega_0$, as:

$$ k_0 = \frac{k_1 + k_2}{2}, \quad \text{and} \quad \omega_0 = \frac{\omega_1 + \omega_2}{2}. $$
And using the identity:

\[
\sin A + \sin B = 2 \sin \frac{1}{2} (A + B) \cos \frac{1}{2} (A - B)
\]  

We can rewrite the amplitude, \( Y \), due to the interaction of the two waves:

\[
Y = y_1 + y_2 = 2A \cos \left[ \frac{(k_1 - k_2)}{2} x - \frac{(\omega_1 - \omega_2)}{2} t \right] \sin \left[ \frac{(k_1 + k_2)}{2} x - \frac{(\omega_1 + \omega_2)}{2} t \right]
\]

16.1.6

Let \( k_1 - k_2 = D_k \), \( \omega_1 - \omega_2 = Dw \), then:

\[
y_1 - y_2 = 2A \cos \frac{1}{2} (D_k x - Dw t) \sin (k_c x - \omega_c t)
\]

16.1.8

which represents two modes of wave propagation, a longer period carrier propagating at \( \omega_0 \) and a higher frequency component of \( Dw \).

Thus the carrier wave propagates at a phase velocity of \( c = \frac{\omega_0}{k_0} \) with an angular frequency, \( \omega_0 = 2\pi f_0 \) modulated by a wave with a frequency, \( \omega_c = 2\pi f_c \). The modulation envelope, or the wave packet, propagates at a group velocity of \( u = \frac{\omega_c}{k} \). Hence \( c \) is different than \( u \) for a dispersive wave.
Summarizing the properties of a dispersive media:

- **Linear dispersionless material**: wave shape remains constant
- **Linear dispersive material**: wave shape changes
- **Nonlinear dispersionless material**: wave shape changes
- **Nonlinear dispersive material**: wave shape remains constant