15.4 Sensitivity and Magnification of Seismometers:

The sensitivity of a seismometer is defined as the ratio of the maximum displacement, $\phi_m$, of the mass to the maximum ground motion, $X_m$, during steady-state motion. The sensitivity is a measure of magnification produced directly by the transducer. It generally ranges from 1 to 10.

Defining the ratio of the driving frequency to the natural frequency:

$$ r = \frac{\Omega_e}{\Omega_o} $$

$\Omega_e$ - angular frequency of the ground or earth, and $\Omega_o$ - the natural frequency of the seismometer.

The normal case for earthquake measurements is for underdampening requiring $\beta < 1$ which gives a total solution:

$$ \phi_{\text{Total}}(t) = \phi_{\text{ss}}(t) + \phi_{\text{TR}}(t) $$

$$ \phi_{\text{Total}}(t) = e^{-\beta \Omega_o t} \left[ c_1 e^{\Omega_o t \sqrt{\beta^2 - 1} + c_2 e^{-\Omega_o t \sqrt{\beta^2 - 1}}} + \frac{r^2 X_o \cos(\Omega_e t + \theta)}{\sqrt{(1 - r^2)^2 + (2\beta r)^2}} \right] $$

The sensitivity is defined by:

$$ \frac{\phi_m}{X_m} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\beta r)^2}} $$

where $X_m = X_o \cos(\Omega t + \theta)$.

Defining the ratio, $r$, of the periods of earth motion to seismometer frequency using:

$$ \Omega_e = 2\pi f_e = \frac{2\pi}{T_e} \quad \text{and} \quad \Omega_o = 2\pi f_o = \frac{2\pi}{T_o} $$

where $T_e$ and $T_o$ are the period of the earth and the seismometer respectively.
It then follows that:

\[
\frac{r}{\Omega_o} = \frac{f_e}{T_o} = \frac{f_c}{T_c}
\]  \hspace{1cm} 15.4.3

and the sensitivity of the seismometer is then:

\[
\frac{\phi_m}{X_m} = \frac{1}{\sqrt{1 - \left(\frac{T_c}{T_o}\right)^2 + \left[2 \beta \left(\frac{T_e}{T_o}\right)\right]^2}}
\]  \hspace{1cm} 15.4.4

Evaluating equation 15.4.4 gives us an insight into the behavior of the seismometer as a function of earth vibration frequency. For example when \( \Omega_c = 0 \) then:

\[
\frac{\phi_m}{X_m} = 0
\]

but when:

\( \Omega_c = \Omega_o \), we see that \( \frac{\phi_m}{X_m} = \frac{1}{2\beta} \).

Thus, zero damping implies unbounded resonant free motion of the seismometer mass required by: \( \Omega_o = \Omega_c \). However, as \( \Omega_c \) becomes large compared to \( \Omega_o \), the ratio \( \frac{\phi_m}{X_m} \), approaches unity indicating that the mass remains stationary (due to its high moment of inertia) while the frame moves with the earth motion.

Now how does the sensitivity vary with various values of \( \Omega_c \). For the case, \( \Omega_c \ll \Omega_o \), which implies \( r < 1 \) then:

\[
\frac{\phi_m}{X_m} \approx \left(\frac{\Omega_c}{\Omega_o}\right)^2
\]  \hspace{1cm} 15.4.5

since
\[
\left(\frac{\Omega_e}{\Omega_o}\right)^2 \ll \frac{\Omega_e}{\Omega_o}
\]

then

\[
\phi_m = \frac{\Omega_e^2}{\Omega_o^2} X_m = \frac{1}{\Omega_o^2} \left(\frac{d^2x}{dt^2}\right)
\]

Hence, the motion of the mass is proportional to \textit{acceleration}. Thus, we can design an accelerometer by requiring that \(\Omega_e < \Omega_o\) by requiring that the natural frequency of the seismometer is much greater than the periods expected from the earth vibrations.

For the case of \(\Omega_e \gg \Omega_o\), then

\[
\frac{\phi_m}{X_m} \approx 1 \quad \text{or} \quad \phi_m = X_m
\]

15.4.7

This is the case of the mass motion being proportional to the \textit{displacement}. Hence, to measure displacement we require that our seismometer period is such that \(\Omega_e \gg \Omega_o\).