15.2 Seismometer Response

First neglect the external driving force and solve the homogeneous part of the differential equation by letting \(\dot{\chi} = 0\). This simulates the transient response of the seismometer and gives:

\[
\ddot{\phi}_t + 2\beta \Omega_0 \dot{\phi}_t + \Omega_0^2 \phi_t = 0 \tag{15.2.1}
\]

which is an equation of admirable reputation for the "damped harmonic oscillator".

Going back to differential equations, we assume a solution to 15.2.1 of the form, \(e^{st}\), with the proper choice of sign:

\[
\phi_t = Ae^{s_1 t} + Be^{s_2 t} = e^{st} \tag{15.2.2}
\]

or in general:

\[
\phi_t = e^{st} \tag{15.2.3}
\]

then differentiating:

\[
\frac{d\phi_t}{dt} = s \ e^{st} = s \phi_t \tag{15.2.4}
\]

\[
\frac{d^2\phi_t}{dt^2} = s^2 \ e^{st} = s^2 \phi_t \tag{15.2.5}
\]

Substituting 15.2.5 into 15.2.1 gives the characteristic or eigenvalue equation.

\[
s^2 \phi_t + 2\beta \Omega_0 s \phi_t + \Omega_0^2 \phi_t = \phi_t \left(s^2 + 2\beta \Omega_0 s + \Omega_0^2\right) = 0 \tag{15.2.6}
\]

Solving for \(s\):
The final transient solution, $f$, is given by substituting 15.2.7 into 15.2.2:

$$s = -2\beta \omega_0 \pm \sqrt{4\beta^2 \omega_0^2 - 4\omega_0^2}$$

or:

$$s = -\beta \omega_0 \pm \omega_0 \sqrt{\beta^2 - 1}$$ \hspace{1cm} 15.2.7

The final transient solution, $f_t$, is given by substituting 15.2.7 into 15.2.2:

$$f_{TR} = A e^{-t \left[ -\beta \omega_0 + \omega_0 \sqrt{\beta^2 - 1} \right]} + B e^{-t \left[ -\beta \omega_0 - \omega_0 \sqrt{\beta^2 - 1} \right]}$$ \hspace{1cm} 15.2.8

where and A and B are constants determined by the initial boundary conditions, i.e. the initial position and initial velocity of the mass. Thus giving the final solution:

$$f_{TR} = e^{-\beta \omega_0 t} \left[ A e^{\beta \omega_0 \sqrt{\beta^2 - 1} t} + B e^{-\omega_0 \sqrt{\beta^2 - 1} t} \right]$$ \hspace{1cm} 15.2.9

However, the solution of $f$ depends greatly upon the value of $b$, the damping constant, i.e., whether $\beta = 1$, $\beta > 1$, or $\beta < 1$. Thus it depends upon a non-linear relationship between, $\beta$, the damping constant and $\omega_0$, the natural undamped frequency of the seismometer.

Case I. When $\beta > 1$, then $\sqrt{\beta^2 - 1}$ is real our solution is for an overdamped oscillator, i.e. with no harmonic motion, i.e. no over- or undershoots of the mass as it returns to the equilibrium position at rest.

![AMPMITUDE](image)

Case II. When $\beta = 1 \rightarrow \sqrt{\beta^2 - 1} = 0$ gives critical damping, here $\phi$ returns to $\phi = 0$ as soon as possible without overshooting. To obtain this case requires a choice of appropriate constants in:
\[ \beta = \frac{\eta}{2\sqrt{km}} \]

and the response looks like:

This is the ideal response because the seismometer output is identical to that of the input, but it is difficult to build such a beast. \( \Phi_{TR} \) then becomes:

\[ \Phi_{TR} = (A+B) e^{\Omega_{0}t} = c e^{-\frac{1}{\sqrt{\beta}}}. \quad 15.2.10 \]

Case III. When \( \beta < 1 \rightarrow \sqrt{\beta^2 - 1} \) is imaginary and the system is said to be underdamped. This produces sinusoidal motion with an exponential decay envelope.

when:
\[ \beta < 1 \left( \sqrt{\beta^2 - 1} \Rightarrow i \sqrt{1 - \beta^2} \right) \]

In this case:

\[ \phi_{TR} = e^{-\beta \Omega o t} \left[ A e^{+t \Omega o \sqrt{1 - \beta^2}} + B e^{-t \Omega o \sqrt{1 - \beta^2}} \right] \]

and we can expand this equation using Euler's expression: \( e^{i\theta} = \cos \theta + i \sin \theta \)

\[ \phi_{TR} = e^{-\beta \Omega o t} \left[ (A+B) \cos (\Omega o \ t \sqrt{1 - \beta^2}) + i (A-B) \sin (\Omega o \ t \sqrt{1 - \beta^2}) \right] \]

Using the identity for the transcendental function for the sum of two angles employs the phase shift of \( \delta \):

\[ \sin (\Omega t + \delta) = \left| \sin \Omega t \cos \delta + \cos \Omega t \sin \delta \right| \]

the solution can be rearranged in the form:

\[ \phi_{TR} = c'e^{-\beta \Omega o t} \ln \left( \Omega o \ t \sqrt{1 - \beta^2} + \delta \right) \]

\( \delta \) is now the phase angle that corresponds to a lead or lag in time between the input and output signals. Obviously our output can not take place before the ground moves, so we have a phase lag in radians that can be converted into time following the actual ground motion.

Generally, we seek to set our seismometers so that they are slightly underdamped, i.e., so there is a slight overshoot of about 1/20.

![Image of a graph showing amplitude over time, with the overshoot set at ~ 1/20.](image-url)

This corresponds to a dampening value of \( \beta \sim 0.6 \) to 0.7 and is the range used in general field seismometers.
Damping properties:

1. Define, $\Omega_f$, the damping frequency as: 
$$\Omega_f = \Omega_0 \sqrt{1 - \beta^2}$$

A plot of $\frac{\Omega_f}{\Omega_0}$ vs $\beta$ gives the value of the degree of damping which looks like:

2. A decay time constant, $\tau$, can be defined where $\tau$ is the time required for the envelope to decay to $1/e$ of its original value:
$$e^{-\beta \Omega_0 \tau} = \frac{1}{e} \quad \text{when} \quad \tau = \frac{1}{\beta \Omega_0}$$

3. Logarithmic decrement, $d$, the logarithm of ratios of successive peaks is frequently used in engineering applications.

$$d = \ln \frac{\phi_1}{\phi_2} = \ln \left( \frac{e^{-\beta \Omega_0 \tau_0}}{e^{-\beta \Omega_0 (\tau_0 + T)}} \right) = \beta \Omega_0 T = \frac{2 \pi \beta}{\sqrt{1 - \beta^2}}$$